# ON HEATING AND MELTING OF A SOLID BODY OWING TO FRICTION

## (O NAGREVANII I PLAVLENII TVERDOGO TELA OT TRENIIA)

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The formulation and solution for the simplest case of two problems dealing with heating and melting of a solid body by friction are given in this article. The first is the problem of friction of a solid body upon a solid body. The second is the problem of the flow around a solid body of a viscous incompressible liquid.

1. 1. If two solid bodies, in contact at some surface and pressed by some force in the direction normal to that surface, are brought into relative motion, while maintaining their contact and mutual compression, they will be heated by friction because of the conversion of mechanical energy into heat. At some particular conditions the amount of heat generated may be sufficient to begin melting one or both of the rubbing bodies. Thus, at the instant when at some contact areas the melting temperature of one of the bodies is reached, the melting front will begin to move toward from that surface; the melting front separates the liquid and solid phases of the material. While the bodies are in relative motion and the liquid phase appears generally speaking - as a viscous liquid, the latter will be drawn into motion. Consequently, the problem must be solved in the following manner. First, it is necessary to solve a simple transient heat transfer problem knowing initial and boundary conditions (the boundary conditions are determined by a given connection between heat flux at contacting areas, which is determined there by the amount of heat generated by friction). If in this solution at some boundary locality, at a point or at a line, the temperature reaches the melting point and in the next instant in that region and in its neighborhood the temperature is above melting, it is necessary, beginning at this instant, to solve the problem by considering the formation of the liquid phase. Indeed, it is necessary to introduce into the investigation the region within which the material is in a molten state and flows. The temperature distribution in the liquid phase then must be found,using the heat conductivity equation of the flowing liquid. The flow of liquid itself must be found with the

use of Navier-Stokes equations for the incompressible fluid, since the molten metal is a liquid which is considered to be incompressible. The boundary between the liquid and solid phases (the melting front) is formed in a point or on a line at the contact area and is not known in advance. It should be found while solving the problem. Definite conditions of compatibility should be satisfied on that boundary; they are due to basic laws, governing the investigated phenomenon and binding together the values of the data looked for on both sides of the melting front. The temperature distribution in the solid phase should be obtained by the use of the ordinary heat conductivity equation.

It should be noted that the problem discussed is - from a physical point of view - a complicated variant of Stefan's classical problem of melting and freezing of an undisturbed medium. The complication consists in the fact that the liquid phase, here, is set in motion; consequently, for the solution of this problem, it is necessary to introduce the hydrodynamics of the viscous liquid. This last condition renders the mathematical problem in principle more complicated than Stefan's problem. Here we will examine the simplest self-similar solution of the problem, which will permit a closed form solution and will manifest the most significant qualitative properties of the phenomenon.

2. The laws of mass and energy conservation and the momentum equation should be satisfied on the melting front. This leads to the following conditions:

$$\begin{gathered}
\rho_1 (D - V_{n1}) = \rho_2 (D - V_{n2}) \\
\rho_1 (D - V_{n1}) V_{n1} + P_{nn1} = \rho_2 (D - V_{n2}) V_{n2} + P_{nn2} \\
\rho_1 (D - V_{n1}) V_{\tau 1} + P_{n\tau 1} = \rho_2 (D - V_{n2}) V_{\tau 2} + P_{n\tau 2} \qquad (1.1) \\
\rho_1 (D - V_{n1}) \left( \frac{V_1^2}{2} + \varepsilon_1 \right) + \bar{P}_{n1} \bar{V}_1 + k_1 \frac{\partial T_1}{\partial n} = \\
= \rho_2 (D - V_{n2}) \left( \frac{V_2^2}{2} + \varepsilon_2 \right) + \bar{P}_{n2} \bar{V}_2 + k_2 \frac{\partial T_2}{\partial n}
\end{gathered}$$

Where $D$ is front velocity along the normal, and the indices $n$ and $\tau$ signify the normal and the tangential component, respectively to the front surface. The letter $\epsilon$ indicates the internal energy of a unit mass; the meaning of other symbols is evident without explanation.

Relationships (1.1) show the possibility of two types of surface discontinuities on which melting takes place. First, there are discontinuities on which stresses become discontinuous. These are shock waves, differing from ordinary ones by the fact that the material in them melts. On these surfaces the temperature, in general, becomes discontinuous. Secondly, there are surfaces on which only the heat flux becomes discontinuous, i.e.

$\partial T/\partial n$ and, possibly, the density. On these surfaces the temperature is continuous and is equal to the melting temperature. Such types of discontinuities can be called weak waves of melting.

3. Let a solid plate in the form of a flat surface be pressed on the boundary of a body which occupies half space and has a uniform given temperature, and let the plate begin to move parallel to itself with a constant velocity $v_0$. We will first solve an ordinary heat transfer problem and will find the conditions at which melting will start. By $f$ we will denote the coefficient of friction between the plate and the body. In a unit of time $fPv_0$ units of heat will flow through a unit area of boundary; here $P$ is the pressure of the plate upon the body (we consider that all the work of friction forces is absorbed by the body only, i.e. the plate is adiabatic). If $P = At^{-1/2}$, the problem will be reduced to integration of a simple equation; the problem thus has a self-similar solution. Indeed, the problem consists in the determination of the solid phase temperature $T_1$ as a solution of equation

$$\frac{\partial T_1}{\partial t} - a_1 \frac{\partial^2 T_1}{\partial x^2} = 0 \tag{1.2}$$

with additional conditions

$$T_1(x, 0) = -T_0 < 0, \qquad \left.\frac{\partial T_1}{\partial x}\right|_{x=0} = -\frac{1}{k_1} fAv_0 t^{-1/2} \tag{1.3}$$

Where $a_1$, $k_1$ are coefficients of diffusivity and thermal conductivity of the body, respectively; for simplicity, we will consider them as well as all subsequent thermal and hydrodynamic coefficients to be constant.* The temperature will be measured from the melting point. The thermal values will be measured in mechanical units.

The direct application of dimensional analysis [1] to problem (1.2), (1.3) does not permit its self-similar solution. However, substituting $T = T_0 u_1$, we see that $u_1$ depends only upon $x$, $t$, $a_1$ and $b = fAv_0/k_1 T_0$ and the application of dimensional analysis in this case shows that

$$u_1 = u_1(\xi, \delta), \qquad \xi = \frac{x}{2\sqrt{a_1 t}}, \qquad \delta = b\sqrt{a_1} = \frac{fAv_0\sqrt{a_1}}{k_1 T_0} \tag{1.4}$$

We obtain the final solution of the problem in the form

$$T_1 = T_0\left\{-1 + \delta\frac{\sqrt{\pi}}{2}[1 - \Phi(\xi)]\right\}, \qquad \Phi(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-\zeta^2} d\zeta \tag{1.5}$$

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* Assuming that some of these coefficients are functions of temperature, which actually is the case, we will not violate the self-similar characters of the solution of the problems being solved here; however, we will lose the possibility of obtaining the solutions in closed form.

The temperature of the boundary of the body is

$$T_1(0, t) = T_0\left(-1 + \delta\frac{\sqrt{\pi}}{2}\right)$$

Therefore, melting will not occur, as long as this temperature remains lower than the melting temperature, i.e. negative.

Consequently, melting will not take place, if $\delta < \delta_* = 2/\sqrt{\pi}$ and it must take place at $\delta > \delta_*$. Here and later, the asterisk indicates critical values. Therefore, at $\delta > \delta_*$ the problem should be solved considering the formation of a molten zone.

4. At $\delta > \delta_*$, already at the initial moment, a weak wave of melting will progress from the boundary of the body inward; its law of motion $x = x_0(t)$ is unknown. In the region $0 < x < x_0(t)$ we will have

$$\frac{\partial v}{\partial t} - \nu\frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{Navier-Stokes equation} \tag{1.6}$$

$$\frac{\partial T_2}{\partial t} - a_2\frac{\partial^2 T_2}{\partial x^2} = \frac{\nu}{c_{p_2}}\left(\frac{\partial v}{\partial x}\right)^2 \quad \text{energy equation} \tag{1.7}$$

Here $c_{p_2}$ is the specific heat of molten material. In the region $x_0(t) < x < \infty$

$$\frac{\partial T_1}{\partial t} - a_1\frac{\partial^2 T_1}{\partial x^2} = 0 \tag{1.8}$$

Additional conditions, obtained with consideration of (1.1), are

$$\begin{gathered} v(0, t) = v_0, \qquad v(x_0, t) = 0, \qquad \left.\frac{\partial T_2}{\partial x}\right|_{x=0} = 0 \\ T_2(x_0, t) = T_1(x_0, t) = 0 \qquad (T_2(x, t) \geqslant 0,\ T_1(x, t) \leqslant 0) \\ T_1(x, 0) = -T_0, \qquad k_1\left.\frac{\partial T_1}{\partial x}\right|_{x=x_0} - k_2\left.\frac{\partial T_2}{\partial x}\right|_{x=x_0} = q\rho\frac{dx_0}{dt} \end{gathered} \tag{1.9}$$

where $q$ is the heat of fusion of unit mass.

It should be noted that for simplicity we assume that the melting temperatures are independent of the pressure, and that the density does not vary during the transfer from one to the other phase.

As above, direct use of dimensional analysis for problem (1.6)-(1.9) leads to the conclusion that it has no self-similar solutions. However, by substituting $v = v_0V$, $T_1 = T_0U_1$, $T_2 = T_0U_2$ and dividing the last of the (1.9) relations by $k_1$, we see that the dimensionless functions $V$, $U_1$, $U_2$ depend upon $x$, $t$, $\nu$, $a_1$, $a_2$, $v_0^2/T_0c_{p_2}$, $\kappa = k_2/k_1$ $q\rho/k_1T_0$, consequently, in accordance with the $\Pi$-theorem of dimensional analysis this relation is reduced to relations depending upon

$$\xi = \frac{x}{2\sqrt{a_1 t}}, \quad \lambda = \frac{\nu}{a_1}, \quad \alpha = \frac{a_2}{a_1}, \quad n = \frac{2q\rho a_1}{k_1 T_0}, \quad m = \frac{v_0^2}{T_0 c_{p_2}}, \quad \varkappa = \frac{k_2}{k_1} \tag{1-10}$$

i.e. the problem in fact has a self-similar solution. The law of motion of a weak melting wave is determined up to a constant $\xi_0$ by the formula

$$x_0(t) = 2\xi_0 \sqrt{a_1 t} \tag{1.11}$$

The unknown constant $\xi_0$ must be found during the course of the solution of the problem. The solution of the problem (1.6)-(1.9), which does not yet satisfy the last of the conditions (1.9), can be easily found in the following form

$$v = v_0 V = v_0 \left[1 - \frac{\Phi(\xi/\sqrt{\lambda})}{\Phi(\xi_0/\sqrt{\lambda})}\right] \qquad (0 \leqslant \xi \leqslant \xi_0) \tag{1.12}$$

$$T_2 = T_0 U_2 = T_0 \frac{\lambda m \sqrt{\pi}}{2\sqrt{\alpha}} \left\{ \int_0^{\xi_0} [\Phi(\xi_0/\sqrt{\alpha}) - \Phi(\tau/\sqrt{\alpha})] \left(\exp\frac{\tau^2}{\alpha}\right) [V'(\tau)]^2 \, d\tau - \right.$$

$$\left. - \int_0^{\xi} [\Phi(\xi/\sqrt{a}) - \Phi(\tau/\sqrt{a}) \left(\exp\frac{\tau^2}{a}\right) [V'(\tau)]^2 \, d\tau \right\} \qquad (0 \leqslant \xi \leqslant \xi_0) \tag{1.13}$$

$$T_1 = T_0 U_1 = -T_0 \frac{\Phi(\xi) - \Phi(\xi_0)}{1 - \Phi(\xi_0)} \qquad (\xi_0 \leqslant \xi \leqslant \infty) \tag{1.14}$$

In these formulas $\xi_0$ is still an unknown parameter. It should be found, by satisfying the last of the conditions (1.9). Inserting into that condition functions (1.12)-(1.14), we obtain a transcendental equation for $\xi_0$

$$\frac{\varkappa m}{\alpha} \left[\exp\left(-\frac{\xi_0^2}{a}\right)\right] \frac{\Phi_{\alpha,\lambda}(\xi_0)}{[\Phi(\xi_0/\sqrt{\lambda})]^2} - \frac{\exp(-\xi_0^2)}{1 - \Phi(\xi_0)} = \frac{\sqrt{\pi}}{2} n\xi_0 \tag{1.15}$$

where

$$\Phi_{\alpha,\lambda}(\xi_0) = \frac{2}{\sqrt{\pi}} \int_0^{\xi_0} \exp\left[\left(\frac{1}{\alpha} - \frac{2}{\lambda}\right)\tau^2\right] d\tau \tag{1.16}$$

The study of this equation shows that it has at any $\kappa > 0$, $a > 0$, $\lambda > 0$, $m > 0$ and $n > 0$ a single root $\xi_0 > 0$. If this root is small, it is approximately expressed by the formula

$$\xi_0 \approx \frac{\varkappa\lambda m\sqrt{\pi}}{2\alpha} = \frac{\sqrt{\pi}}{2} \frac{k_2 \nu v_0^2}{k_1 a_2 c_{p2} T_0} \ll 1 \tag{1.17}$$

The small value of $\xi_0$ signifies a relatively thin molten layer, a slow melting. The expression (1.17) indicates that this takes place at a relatively low velocity $v_0$, low viscosity $\nu$ or low heat transfer coefficient of the liquid phase $k_2$ and also at relatively large heat transfer coefficient of the solid phase $k_1$ or large diffusivity of the liquid phase $a_2$

or large liquid phase specific heat $c_{p_2}$ or at a very low initial temperature of the medium $T_0$. All this fully conforms to physical intuition.

It should be also noted that the right-hand side of formula (1.17) does not contain the parameter $n$, i.e. that if the parameter $\xi_0$ is small, it does not depend upon the heat of fusion $q$. This means that the law of motion of the melting front at $\xi_0 \ll 1$ does not depend upon the heat of fusion.

The solution constructed above is applicable at condition $\delta > \delta_*$, i.e. at

$$\frac{fAv_0\sqrt{a_1}}{k_1T_0} > \frac{2}{\sqrt{\pi}} \tag{1.18}$$

In partirular, if all physical parameters of the medium and the magnitudes of $A$, $T_0$ are fixed, expression (1.18) determines the lower limit for the velocity $v_0$, beyond which melting will begin. In fact, melting does not occur if $v_0 < v_0^*$, and it takes place if $v_0 > v_0^*$, where

$$v_0^* = \frac{2}{\sqrt{\pi}}\frac{k_1T_0}{fA\sqrt{a_1}} \tag{1.19}$$

We note especially, that if $v_0$ is increased, while going through $v_0^*$, then $\xi_0$ begins to alter not from the value $\xi_0 = 0$, but from some finite value $\xi_{0\,\min}$, i.e. that no matter how little $v_0$ exceeds $v_0^*$, the melting front will displace itself with a $\xi_0$ different from zero. In order to find $\xi_{0\,\min}$ in the solution of equation (1.15), which has the form: $\xi_0 = \xi_0(\lambda, a, \kappa, m, n)$, one must substitute its minimum value for $m$, i.e.

$$m_{\min} = m^* = (v_0^*)^2 / T_0c_{p2}$$

At small values of $\xi_0$ we obtain an approximate expression for $\xi_{0\,\min}$, using (1.17), (1.19):

$$\xi_{0\,\min} \approx \frac{2}{\sqrt{\pi}}\frac{k_1k_2\nu T_0}{a_2a_1f^2A^2c_{p2}} \tag{1.20}$$

5. In the paper [2], for the determination of the coefficient of friction between the surface of a projectile (penetrating at high velocity through an armor plate) and the surface of the armor, the author assumes that the temperature at the contact surface increases rapidly owing to generation of a large amount of heat, reaches the melting point and is then maintained at that level. This assumption enables us to solve a transient heat transfer problem for the armor provided that the temperature at the boundary is equal to the melting temperature. Utilizing this solution, the author finds the coefficient of friction and determines the fraction of the projectile kinetic energy loss due to friction against the armor plate. The solution developed here by us shows that the initial assumption used by the author of paper [2], strictly speaking, is incorrect: no

matter how small the viscosity of the liquid phase of the medium is, at conditions when the boundary temperature must exceed the melting temperature, this temperature can not be maintained at the melting point, and melting is bound to begin. However, if the viscosity of the liquid phase is relatively small, then, as the formula (1.17) shows, melting will take place slowly.

Melting will never occur at any conditions if the viscosity is equal to zero. In this case at $v_0 > v_0{}^*$ the solution of our problem is formed in the following manner. Besides the condition (1.3) at the boundary, the condition $T_1(0,\ t) = 0$ is also given; i.e. the boundary temperature is assumed to be equal to the melting temperature. Since the problem thus becomes indeterminate, it is necessary to consider that the coefficient of friction $f$ is no longer equal to the coefficient of dry friction, but is an unknown value determined from the solution of the problem.

Physically this is explained by the fact that at these conditions the boundary is no longer solid, since the temperature on it is equal to the melting temperature, but it is not liquid either, as long as the medium can not melt, provided the liquid phase is an ideal liquid. The boundary is in a semi-liquid state such that the coefficient of friction on it should have a value prescribed by the condition of the problem. In the case discussed it is determined by the formula

$$f^* = \frac{2}{\sqrt{\pi}} \frac{k_1 T_0}{v_0 A \sqrt{a_1}} = f \frac{v_0{}^*}{v_0} < f \tag{1.21}$$

where $f$ is the coefficient of friction.

Consequently, the assumption in the paper [2] will be true under the condition that the viscosity of the melted armor is equal to zero; and if it is small, this assumption then can be regarded as an approximate boundary condition and the results of the paper [2] can be regarded as approximate.

It should be also noted that it is possible similarly to solve the friction problem of two bodies occupying two half spaces and to determine when melting of these bodies will begin, and which will begin melting first. Here, however, we do not give the solution of this problem.

2. If a solid body begins to move within a viscous liquid (gas), then on account of viscous energy dissipation in the moving liquid and of the heat transfer coefficient of the liquid and of the body, the latter (as well as the liquid) will be heated. At certain conditions this heating may lead to melting of the body. For a quantitative description of this phenomenon, it is necessary to solve the problem of the motion of the fluid around the body, the motion of the melted phase and the temperature distribution in the liquid and in the body. The melting waves thus created

should here be regarded as weak ones. The mathematical difficulties associated with the solution of this problem are exceedingly great. The problem here is considered in the simplest self-similar formulation; and in its simple form the solution obtained will reveal some qualitative features of the phenomena.

Let a solid body, occupying a half space, bounded with viscous incompressible fluid (filling the rest of the space) begin to move with a constant velocity $v_0$ parallel to the boundary. We will denote the initial temperatures of the fluid and of the body by $T_{10}$ and $T_{30}$, respectively, the density and the viscosity of the liquid by $\rho_1$, $\nu_1$, the thermal conductivity and diffusivity by $k_1$, $a_1$; for the liquid phase of the body we will use also $\rho_1$, $\nu_2$, $k_2$. $a_2$, and for the solid phase we will use $k_3$ and $a_3$ for the thermal conductivity and diffusivity. We will denote by $q$ the heat of fusion. For simplicity we consider that all these coefficients are constant, and that the density of the body does not vary during melting. The problem then is reduced to the evaluation of functions

$$v_1, \quad v_2, \quad z_1 = c_{p1}T_1, \quad z_2 = c_{p2}T_2, \quad z_3 = c_{p3}T_3$$

(where $v_1$, $v_2$ are velocities of the flowing liquid and of the liquid phase; $T_1$, $T_2$, $T_3$, corresponding temperatures and $c_{pi}$ specific heat) from relationships

$$\frac{\partial v_1}{\partial t} - \nu_1 \frac{\partial^2 v_1}{\partial x^2} = 0, \qquad \frac{\partial v_2}{\partial t} - \nu_2 \frac{\partial^2 v_2}{\partial x^2} = 0$$

$$\frac{\partial z_1}{\partial t} - a_1 \frac{\partial^2 z_1}{\partial x^2} = \nu_1 \left(\frac{\partial v_1}{\partial x}\right)^2, \qquad \frac{\partial z_2}{\partial t} - a_2 \frac{\partial^2 z_2}{\partial x^2} = \nu_2 \left(\frac{\partial v_2}{\partial x}\right)^2$$

$$\frac{\partial z_3}{\partial t} - a_3 \frac{\partial^2 z_3}{\partial x^2} = 0$$

$$v_1(x, 0) = v_0, \quad z_1(x, 0) = z_{10}, \quad z_3(x, 0) = z_{30} \tag{2.1}$$

$$v_1(-\infty, t) = v_0, \quad v_1(0, t) = v_2(0, t), \quad \rho_1\nu_1 \frac{\partial v_1}{\partial x}\bigg|_{x=0} = \rho_2\nu_2 \frac{\partial v_2}{\partial x}\bigg|_{x=0}$$

$$v_2(x_0, t) = 0, \quad z_1(-\infty\ t) = z_{10}, \quad \frac{1}{c_{p1}} z_1(0, t) = \frac{1}{c_{p2}} z_2(0, t)$$

$$\frac{1}{c_{p1}} k_1 \frac{\partial z_1}{\partial x}\bigg|_{x=0} = \frac{1}{c_{p2}} k_2 \frac{\partial z_2}{\partial x}\bigg|_{x=0}, \quad \frac{1}{c_{p2}} z_2(x_0, t) = \frac{1}{c_{p3}} z_3(x_0, t) = \frac{1}{c_{p3}} z_*$$

$$z_3(\infty, t) = z_{30}, \quad \frac{1}{c_{p3}} k_3 \frac{\partial z_3}{\partial x}\bigg|_{x=x_0} - \frac{1}{c_{p2}} k_2 \frac{\partial z_2}{\partial x}\bigg|_{x=x_0} = q\rho_2 \frac{dx_0}{dt}$$

Here $x_0 = x_0(t)$ is the law of motion of a weak melting wave which also should be determined.

As in Section 1, it is shown that the problem has a self-similar solution and that the dimensionless functions

$$V_1=\frac{v_1}{v_0},\quad V_2=\frac{v_2}{v_0},\quad U_1=\frac{z_1}{z_{30}},\quad U_2=\frac{z_2}{z_{30}},\quad U_3=\frac{z_3}{z_{30}}$$

depend upon

$$\begin{aligned}
&\xi=\frac{x}{2\sqrt{a_3t}},\ \lambda_1=\frac{\nu_1}{a_3},\ \lambda_2=\frac{\nu_2}{a_3}, && \alpha_1=\frac{a_1}{a_3}\\
&\alpha_2=\frac{a_2}{a_3},\quad r=\frac{\rho_1}{\rho_2},\ \theta=\frac{z_{10}}{z_{30}}, && \theta_*=\frac{z_*}{z_{30}}\\
&\varkappa_1=\frac{k_1}{k_3},\quad \varkappa_2=\frac{k_2}{k_3},\ n=\frac{2q\rho_2a_3c_{p3}}{z_{30}k_3}, && m=\frac{v^2{}_0}{z_{30}}\\
&\varphi_1=\frac{c_{p1}}{c_{p3}},\quad \varphi_2=\frac{c_{p2}}{c_{p3}}
\end{aligned}\tag{2.2}$$

Here $z_*=c_{p3}T_*$ is the "temperature" of melting.

The law of motion of the melting front is determined by the expression

$$x_0=2\xi_0\sqrt{a_3t}\tag{2.3}$$

where $\xi_0$ is the constant which should be determined. The solution is given by the expressions

$$\begin{aligned}
&V_1=1-a[1+\Phi(\xi/\sqrt{\lambda_1})] && (-\infty\leqslant\xi\leqslant0)\\
&V_2=1-a-b\Phi(\xi/\sqrt{\lambda_2}) && (0\leqslant\xi\leqslant\xi_0)
\end{aligned}\tag{2.4}$$

$$U_1=C_1+C_2\Phi(\xi/\sqrt{\alpha_1})-\frac{\lambda_1m}{\alpha_1}\int_0^{\xi}\left(\exp\left(-\frac{\tau^2}{\alpha_1}\right)\right)\left(\int_0^{\tau}\left(\exp\frac{\zeta^2}{\alpha_1}\right)[V_1'(\zeta)]^2\,d\zeta\right)d\tau$$

$$(-\infty\leqslant\xi\leqslant0)$$

$$U_2=A_1+A_2\Phi(\xi/\sqrt{\alpha_2})-\frac{\lambda_2m}{\alpha_2}\int_0^{\xi}\left[\exp\left(-\frac{\tau^2}{\alpha_2}\right)\right]\left(\int_0^{\tau}\left(\exp\frac{\zeta^2}{\alpha_2}\right)[V_2'(\zeta)]^2\,d\zeta\right)d\tau$$

$$(0\leqslant\xi\leqslant\xi_0)$$

$$U_3=1-(1-\theta_*)\frac{1-\Phi(\xi)}{1-\Phi(\xi_0)}\qquad(\xi_0\leqslant\xi\leqslant\infty)$$

where the constants $a$, $b$, $C_1$, $C_2$, $A_1$, $A_2$ are found from the relationships

$$\frac{1}{a} = 1 + r\sqrt{\frac{\lambda_1}{\lambda_2}}\Phi(\xi_0/\sqrt{\lambda_2}), \qquad \frac{1}{b} = \frac{1}{r}\sqrt{\frac{\lambda_2}{\lambda_1}}\left[1 + r\sqrt{\frac{\lambda_1}{\lambda_2}}\Phi(\xi_0/\sqrt{\lambda_2})\right]$$

$$C_1 - C_2 - \frac{\lambda_1 m}{\alpha_1}\int_0^{-\infty}\left[\exp\left(-\frac{\tau^2}{\alpha_1}\right)\right]\left(\int_0^{\tau}(\exp\frac{\zeta^2}{\alpha_1})[V_1'(\zeta)]^2 d\zeta\right)d\tau = \theta \qquad (2.5)$$

$$A_1 + A_2\Phi(\xi_0/\sqrt{\alpha_2}) - \frac{\lambda_2 m}{\alpha_2}\int_0^{\xi_*}\left[\exp\left(-\frac{\tau^2}{\alpha_2}\right)\right]\left(\int_0^{\tau}(\exp\frac{\zeta^2}{\alpha_2})[V_2'(\zeta)]^2 d\zeta\right)d\tau = \varphi_2\theta_*$$

$$C_1\varphi_2 = A_1\varphi_1, \qquad \frac{\varphi_2\varkappa_1 C_2}{\sqrt{\alpha_1}} = \frac{\varphi_1\varkappa_2 A_2}{\sqrt{\alpha_2}}$$

and the parameter $\xi_0$ is obtained from the transcendental equation

$$\frac{\sqrt{\pi}}{2}n\xi_0 + \frac{(\theta_* - 1)\exp(-\xi_0^2)}{1 - \Phi(\xi_0)} + \frac{\varkappa_2}{\varphi_2}\left[\frac{A_2}{\sqrt{\alpha_2}}\exp\left(-\frac{\xi_0^2}{\alpha_2}\right) - \right.$$

$$\left. - \frac{2mb^2}{\sqrt{\pi}\alpha_2}\left[\exp\left(-\frac{\xi_0^2}{\alpha_2}\right)\right]\int_0^{\xi_*}\exp\left[\left(\frac{1}{\alpha_2} - \frac{2}{\lambda_2}\right)\tau^2 d\tau\right]\right] = 0 \qquad (2.6)$$

Let us find the limit for velocity $v_0$, above which melting begins. Solving the problem with the assumption that there is no melting, and requiring the condition of body temperature equal to the melting temperature, we obtain the equation

$$\frac{(\theta_* - 1)\sqrt{\alpha_1}\varphi_1}{\varkappa_1} = \theta - \varphi_1\theta_* + \frac{4m_*}{\pi\alpha_1}J(\alpha_1, \lambda_1)$$

$$J(\alpha_1, \lambda_1) = \int_0^{-\infty}\left[\exp\left(-\frac{\tau^2}{\alpha_1}\right)\right]\left(\int_0^{\tau}\exp\left[\left(\frac{1}{\alpha_1} - \frac{2}{\lambda_1}\right)\zeta^2\right]d\zeta\right)d\tau \qquad (2.7)$$

from which $m_* = (v_{0*})^2/z_{30}$ is found, and consequently also $v_{0*}$ is the critical velocity. The same equation is obtained from equation (2.6) if in the latter we take $\xi_0 = 0$ which means that by increasing $v_0$ and going through $v_{0*}$, $\xi_0$ will begin to increase from $\xi_0 = 0$, which is different from the problem of the preceding paragraph.

From the equation (2.7) we obtain

$$m_* = \frac{\pi\alpha_1}{4J(\alpha_1, \lambda_1)}\left[\frac{\varphi_1\sqrt{\alpha_1}}{\varkappa_1}(\theta_* - 1) - (\theta - \varphi_1\theta_*)\right] \qquad (2.8)$$

or

$$v_{0*}^2 = \frac{\pi}{4}\frac{c_{p1}a_1}{a_3 J\left(\frac{a_1}{a_3}, \frac{\nu_1}{a_3}\right)}\left[\frac{k_3}{k_1}\sqrt{\frac{a_1}{a_3}}(T_* - T_{30}) - (T_{10} - T_*)\right] \qquad (2.9)$$

The condition that $v_{0*}^2 > 0$ is given in view of $J(a_1, \lambda_1) > 0$ by the inequality

$$\frac{k_3}{k_1}\sqrt{\frac{a_1}{a_3}}(T_* - T_{30}) > T_{10} - T_* \tag{2.10}$$

The melting will take place at any velocity, including $v_0 = 0$, if this condition is not achieved. In the latter case melting will take place owing to high initial temperature of the liquid ($T_{10} > T_*$).

It is possible to show that at $v_0 > v_{0*}$ the equation (2.6) has a unique root. Finally, we note, that since $J(a_1, \lambda_1) \to 0$ as $\nu_1 \to 0$, from the formula (2.9) at condition (2.10) we obtain $v_{0*} \to \infty$, which is natural from physical considerations.

Finally, in solving the problem discussed, the basic equation of the flow around elongated bodies can be somewhat simplified. This is done in the theory of an ordinary boundary layer, since the basic causes permitting the corresponding simplifications to be performed also remain in force here. This leads to the development of the corresponding theory of the boundary layer with melting.

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